Solutions for Homework 2

1. The isotropic point kernel and the anisotropic point kernels are used to do problems a and b, respectively.

   1a) The differential length for the ring source is $d\ell = r_0\, d\Phi$.

$$\phi(z_0) = \int_0^{2\pi} \frac{q_0}{4\pi(r_0^2 + z_0^2)}\, r_0\; d\Phi = \frac{q_0 r_0}{2(r_0^2 + z_0^2)}\;.$$

   1b) An extra factor of μ is present in the kernel because of the μ-dependence of the source.

$$\phi(z_0) = \int_0^{2\pi} \frac{q_0}{4\pi(r_0^2 + z_0^2)}\,\frac{z_0}{(r_0^2 + z_0^2)^{1/2}}\, r_0\, d\Phi = \frac{q_0 r_0 z_0}{2(r_0^2 + z_0^2)^{3/2}}\;.$$

2. The isotropic point kernel and the anisotropic point kernels are used to do problems a and b, respectively.

   2a) The differential surface for the disk source is $dA = r\, dr\, d\Phi$.

$$\begin{aligned}\phi(z_0) &= \int_0^{2\pi}\int_0^{r_0} \frac{q_0}{4\pi(r^2 + z_0^2)}\, r\, dr\, d\Phi \\ &= \frac{q_0}{4}\ell n\left\{1 + \frac{r_0^2}{z_0^2}\right\}\;.\end{aligned}$$

2b) An extra factor of μ is present in the kernel because of the μ-dependence of the source.

$$\begin{aligned}\phi(z_0) &= \int_0^{2\pi}\int_0^{r_0} \frac{q_0}{4\pi(r^2+z_0^2)}\,\frac{z_0}{(r^2+z_0^2)^{1/2}}\, r\,dr\,d\Phi \\ &= \frac{q_0}{2}\left\{1-\frac{z_0}{\sqrt{r_0^2+z_0^2}}\right\}.\end{aligned}$$

3a) The limit as $r_0 \to \infty$ of $\frac{q_0}{4}\ell n\left\{1+\frac{r_0^2}{z_0^2}\right\}$ is ∞.

3b) The limit as $r_0 \to \infty$ of $\frac{q_0}{2}\left\{1-\frac{z_0}{\sqrt{r_0^2+z_0^2}}\right\}$ is $\frac{q_0}{2}$.

4a) First we use delta-function sources with vacuum boundary conditions.

a) The equation to be solved is

$$\mu\frac{\partial\psi}{\partial x} = \frac{q_0}{4\pi}\delta(x)\,, \quad \text{for } \mu > 0,$$

on the interval, $[0,\infty)$, with $\psi(0) = 0$. Dividing the above equation by μ and integrating it over $(0-\epsilon, 0+\epsilon)$, where ϵ is arbitrarily small, one finds that the solution must jump at $x = 0$ by $\frac{q_0}{4\pi\mu}$. Thus the angular flux solution is

$$\psi(x) = \frac{q_0}{4\pi\mu}\,.$$

It is clear that scalar flux associated with this angular flux solution is infinite.

b) The equation to be solved is

$$\mu\frac{\partial\psi}{\partial x} = \frac{q_0\mu}{4\pi}\delta(x)\,, \quad \text{for } \mu > 0,$$

on the interval, $[0, \infty)$, with $\psi(0) = 0$. Dividing the above equation by μ and integrating it over $(0 - \epsilon, 0 + \epsilon)$, where ϵ is arbitrarily small, one finds that the solution must jump at $x = 0$ by $\frac{q_0}{4\pi}$. Thus the angular flux solution is

$$\psi(x) = \frac{q_0}{4\pi} \,,$$

and the scalar flux solution is

$$\phi = 2\pi \int_0^1 \frac{q_0}{4\pi} \, d\mu = \frac{q_0}{2} \,.$$

4b) Next we use incident fluxes with zero sources.

a) The equation to be solved is

$$\mu \frac{\partial \psi}{\partial x} = 0 \,, \quad \text{for } \mu > 0,$$

on the interval, $[0, \infty)$. This equation obviously has a constant solution. Remembering that, in general, one divides a surface source by $\overrightarrow{\Omega} \cdot \overrightarrow{n}$ to obtain the corresponding incident flux, we divide $\frac{q_0}{4\pi}$ by μ to obtain the incident flux, which is also the solution:

$$\psi(x) = \frac{q_0}{4\pi\mu} \,.$$

As previously noted, the corresponding scalar flux is infinite.

a) The equation to be solved is

$$\mu \frac{\partial \psi}{\partial x} = 0 \,, \quad \text{for } \mu > 0,$$

on the interval, $[0, \infty)$. We divide $\frac{q_0 \mu}{4\pi}$ by μ to obtain the incident flux, which is also the solution:

$$\psi(x) = \frac{q_0}{4\pi} \, .$$

As previously noted, the corresponding scalar flux solution is

$$\phi = \frac{q_0}{2} \, .$$

5) Starting with the integral equation for the angular flux,

$$\psi(\overrightarrow{r}, \overrightarrow{\Omega}) = \psi(\overrightarrow{r} - s_b \overrightarrow{\Omega}, \overrightarrow{\Omega}) \exp\left[- \int_0^{s_b} \sigma_t(s') \, ds' \right] + \int_0^{s_b} \mathcal{Q}(\overrightarrow{r} - s \overrightarrow{\Omega}, \overrightarrow{\Omega}) \exp\left[- \int_0^{s} \sigma_t(s') \, ds' \right] \, ds \, , \tag{1}$$

we derived various kernels for the scalar flux in class. This was basically done by integrating Eq. (1) over all directions and manipulating the integrand. The final result was

$$\begin{aligned} \phi(\overrightarrow{r}) \;\; = \;\; & \oint_\Gamma \frac{\psi(\overrightarrow{r}', \overrightarrow{\Omega}_0) |\overrightarrow{\Omega}_0 \cdot \overrightarrow{n}|}{\| \overrightarrow{r}' - \overrightarrow{r} \|^2} \exp\left[-\tau(\overrightarrow{r}', \overrightarrow{r}) \right] \, dA' + \\ & \int_{\mathcal{D}} \frac{\mathcal{Q}(\overrightarrow{r}'', \overrightarrow{\Omega}_0)}{\| \overrightarrow{r}'' - \overrightarrow{r} \|^2} \exp\left[-\tau(\overrightarrow{r}'', \overrightarrow{r}) \right] \, dV'' \, , \end{aligned} \tag{2}$$

where

$$\overrightarrow{\Omega}_0 = \frac{\overrightarrow{r} - \overrightarrow{r}_0}{\| \overrightarrow{r} - \overrightarrow{r}_0 \|} \, , \tag{3}$$

with $\overrightarrow{r}_0$ denoting the integration variable associated with the kernel, e.g., $\overrightarrow{r}_0 = \overrightarrow{r}'$ in the area kernel and $\overrightarrow{r}_0 = \overrightarrow{r}''$ in the volumetric kernel. Since we want $\overrightarrow{J} \cdot \overrightarrow{n}_0$, where $\overrightarrow{n}_0$ is an arbitrary normal, we need simply multiply Eq. (1) by $\overrightarrow{\Omega} \cdot \overrightarrow{n}_0$ before we perfom the angular integration:

$$\psi(\overrightarrow{r}, \overrightarrow{\Omega}) \left(\overrightarrow{\Omega} \cdot \overrightarrow{n}_0\right) = \psi(\overrightarrow{r} - s_b \overrightarrow{\Omega}, \overrightarrow{\Omega}) \left(\overrightarrow{\Omega} \cdot \overrightarrow{n}_0\right) \exp\left[-\int_0^{s_b} \sigma_t(s')\, ds'\right] +$$
$$\int_0^{s_b} \mathcal{Q}(\overrightarrow{r} - s\,\overrightarrow{\Omega}, \overrightarrow{\Omega}) \left(\overrightarrow{\Omega} \cdot \overrightarrow{n}_0\right) \exp\left[-\int_0^{s} \sigma_t(s')\, ds'\right]\, ds\,. \tag{4}$$

Carrying out the very same manipulations on Eq. (4) as were originally carried out on Eq. (1), we get

$$\begin{aligned} \overrightarrow{J}(\overrightarrow{r}) \cdot \overrightarrow{n}_0 \quad = \quad & \oint_\Gamma \frac{\psi(\overrightarrow{r}', \overrightarrow{\Omega}_0) |\overrightarrow{\Omega}_0 \cdot \overrightarrow{n}| \left(\overrightarrow{\Omega}_0 \cdot \overrightarrow{n}_0\right)}{\|\overrightarrow{r}' - \overrightarrow{r}\|^2} \exp\left[-\tau(\overrightarrow{r}', \overrightarrow{r})\right]\, dA' + \\ & \int_\mathcal{D} \frac{\mathcal{Q}(\overrightarrow{r}'', \overrightarrow{\Omega}_0) \left(\overrightarrow{\Omega}_0 \cdot \overrightarrow{n}_0\right)}{\|\overrightarrow{r}'' - \overrightarrow{r}\|^2} \exp\left[-\tau(\overrightarrow{r}'', \overrightarrow{r})\right]\, dV''\,. \end{aligned} \tag{5}$$

6a) For this problem, $\overrightarrow{\Omega}_0 \cdot \overrightarrow{n}_0 = \mu$, so there is just another factor of μ in the integrand:

$$J_z(z_0) = \int_0^{2\pi} \frac{q_0}{4\pi(r_0^2 + z_0^2)} \frac{z_0}{(r_0^2 + z_0^2)^{1/2}}\, r_0\, d\Phi = \frac{q_0 r_0 z_0}{2(r_0^2 + z_0^2)^{3/2}}\,.$$

Note that the current for an isotropic source is the same as the scalar flux for a cosine-law source (see Problem 1b).

6b) For this problem, $\overrightarrow{\Omega}_0 \cdot \overrightarrow{n}_0 = \mu$, so there is just another factor of μ in the integrand:

$$J_z(z_0) = \int_0^{2\pi} \frac{q_0}{4\pi(r_0^2 + z_0^2)} \frac{z_0^2}{(r_0^2 + z_0^2)}\, r_0\, d\Phi = \frac{q_0 r_0 z_0^2}{2(r_0^2 + z_0^2)^2}\,.$$

7a) For this problem, $\overrightarrow{\Omega}_0 \cdot \overrightarrow{n}_0 = \mu$, so there is just another factor of μ in the integrand:

$$\begin{aligned} J_z(z_0) &= \int_0^{2\pi} \int_0^{r_0} \frac{q_0}{4\pi(r^2+z_0^2)} \frac{z_0}{(r^2+z_0^2)^{1/2}}\, r\, dr\, d\Phi \\ &= \frac{q_0}{2}\left\{1 - \frac{z_0}{\sqrt{r_0^2+z_0^2}}\right\}. \end{aligned}$$

Note that the current for an isotropic source is the same as the scalar flux for a cosine-law source (see Problem 2b).

7b) For this problem, $\overrightarrow{\Omega}_0 \cdot \overrightarrow{n}_0 = \mu$, so there is just another factor of μ in the integrand:

$$\begin{aligned} J_z(z_0) &= \int_0^{2\pi} \int_0^{r_0} \frac{q_0}{4\pi(r^2+z_0^2)} \frac{z_0^2}{(r^2+z_0^2)}\, r\, dr\, d\Phi \\ &= \frac{q_0}{4}\left\{1 - \frac{z_0^2}{r_0^2+z_0^2}\right\}. \end{aligned}$$

8a) The limit as $r_0 \to \infty$ of $\frac{q_0}{2}\left\{1 - \frac{z_0}{\sqrt{r_0^2+z_0^2}}\right\}$ is $\frac{q_0}{2}$.

8b) The limit as $r_0 \to \infty$ of $\frac{q_0}{4}\left\{1 - \frac{z_0^2}{r_0^2+z_0^2}\right\}$ is $\frac{q_0}{4}$.